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ON SPECTRAL RESOLUTIONS OF DIFFERENTIAL VECTOR-OPERATORS

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Abstract

We show that spectral resolutions of differential vector-operators may be represented as a specific direct sum integral operator with a kernel written in terms of generalized vector-operator eigenfunctions. Then we prove that a generalized eigenfunction measurable with respect to the spectral parameter may be decomposed using a set of analytical defining systems of coordinate operators.

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1. Preliminaries

1.1. **Historical notes.** In [9], there is an example of a Schrödinger operator generated by the Hamiltonian

$$H = -\frac{d^2}{dx^2} + \left(s^2 - \frac{1}{4}\right)\frac{1}{\cos^2 x}, \quad s > 0. \tag{1}$$

Since the potential of (1) has a countable number of singularities on $\mathbb{R}$ which spoil the local integrability, operators T_i are constructed, generated by (1) in the spaces

$$L^2\left(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi\right), \; i \in \mathbb{Z},$$

and then the direct sum operator $\oplus_{i\in\mathbb{Z}} T_i$ is considered in the space

$$\oplus_{i\in\mathbb{Z}} L^2\left(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi\right).$$

This physical example gave birth to the theory of general differential direct sum operators, or in the text below vector-operators. Beginning from 1992, the theory of differential vector-operators has been investigated in connection with their non-spectral properties in a Hilbert space ([1], [2], [3]) and in complete locally convex spaces ([4], [5]). The interest in such a theory is explained by its numerous applications in theoretical physics and pure mathematics. Thus, physical applications may be found in a single or a multi-particle quantum mechanics, especially in problems where a quantum system is split into a number of disconnected subsystems under the influence of a potential. Such a physical situation is also well described by the theory of Schrödinger operators on graphs and hence under special boundary conditions a differential operator on a graph may be represented as a differential vector-operator. For applications in quantum mechanics see also the respective references in [3]. Some most modern results in connection with the spectral theory of differential operators on graphs may be found in [6, 7, 8] and references therein.

As it was shown in the fundamental works [2] and [3], a differential vector-operator is an object which resembles an ordinary differential operator by its general properties, but in fact it has a much more complicated structure. Only in exceptional cases a differential vector-operator may be interpreted as an ordinary differential operator.

Although the larger part of the studies concerned only non-spectral properties of differential vector-operators, recently there has been some development of their spectral theory. Some results describing the position of spectra of Schrödinger vector-operators were presented in 1985 in [9] and the most recent results for general quasi-differential vector-operators belong to Sobhy El-Sayed Ibrahim [10, 11].

The internal spectral structure of abstract vector-operators was first investigated in [12], for which see also [13, 14]. The structure of coordinate operators as differential operators played the key role in [15] where the isometry making the ordered representation was described in terms of generalized eigenfunctions of a differential vector-operator. In this paper we show that spectral resolutions of differential vector-operators may be represented as a specific direct sum integral operator with a kernel written in terms of generalized vector-operator eigenfunctions (Theorem 2.6). These generalized eigenfunctions appear to be only measurable relative to the spectral parameter, therefore it is an essential problem to obtain their decomposition over some set of analytical kernels. This problem is positively solved by Theorem 2.7 of the current work. Theorem 2.7 in particular gives birth to the spectral theory of differential vector-operators in matrix Hilbert spaces.

1.2. **Mathematical background.** Basic concepts of quasi-differential operators are well described in [2, 3]. A good reference for operators with real coefficients is the book of M.A. Naimark [16].

Let Ω be a finite or a countable set of indices. On Ω, we have a multi-interval differential Everitt-Markus-Zettl system $\{I_i, \tau_i\}_{i\in\Omega}$, where I_i are arbitrary intervals of the real line and τ_i are formally self-adjoint differential expressions of a finite order. This EMZ system generates a family of Hilbert spaces $\{L^2(I_i) = L_i^2\}_{i\in\Omega}$ and families of minimal $\{T_{min,i}\}_{i\in\Omega}$ and maximal $\{T_{max,i}\}_{i\in\Omega}$ differential operators. Consider a respective family $\{T_i\}_{i\in\Omega}$ of self-adjoint extensions. Further, we introduce a system Hilbert space $\mathbf{L}^2 = \oplus_{i\in\Omega} L_i^2$, consisting of the vectors $\mathbf{f} = \oplus_{i\in\Omega} f_i$ such that $f_i \in L_i^2$ and

$$\|\mathbf{f}\|^2 = \sum_{i\in\Omega} \|f_i\|_i^2 = \sum_{i\in\Omega} \int_{I_i} |f_i|^2\, dx < \infty.$$

In the space $\mathbf{L}^2$ consider the operator $T : D(T) \subseteq \mathbf{L}^2 \to \mathbf{L}^2$, defined on the domain

$$D(T) = \left\{ \mathbf{f} \subseteq \mathbf{L}^2 : \sum_{i\in\Omega} \|T_i f_i\|_i^2 < \infty \right\}$$

by $T\mathbf{f} = \oplus_{i\in\Omega} T_i f_i$.

The operator T is called a *self-adjoint differential vector-operator* generated by the self-adjoint extensions T_i, or simply a vector-operator (or shortly a v-operator). If Ω is infinite, the vector-operator T is called *infinite*. The operators T_i are called *coordinate* operators.

The abstract preliminaries for this work may be found, for instance, in books [17, 18].

Fix $i \in \Omega$. For each T_i there exists a unique resolution of the identity E_λ^i and a unitary operator U_i, making the isometrically isomorphic mapping of the Hilbert space L_i^2 onto the space $L^2(M_i, \mu_i)$, where the operator T_i is represented as a multiplication operator. Below, we remind the structure of the mapping U_i.

We call $\phi \in L_i^2$ a *cyclic vector* if for each $z \in L_i^2$ there exists a Borel function f, such that $z = f(T_i)\phi$. Generally, there is no a cyclic vector in L_i^2 but there is a collection $\{\phi^k\}$ of them in L_i^2, such that $L_i^2 = \oplus^k L_i^2(\phi^k)$, where $L_i^2(\phi^k)$ are T_i-invariant subspaces in L_i^2 generated by the cyclic vectors ϕ^k. That is $L_i^2(\phi^k) = \overline{\{f(T_i)\phi^k\}}$, for a varying Borel function f, such that $\phi^k \in D(f(T_i))$.

A vector $\phi \in L_i^2$ is called *maximal* relative to the operator T_i, if each measure $(E^i(\cdot)x, x)_i$, $x \in L_i^2$, is absolutely continuous with respect to the measure $(E^i(\cdot)\phi, \phi)_i$.

For each Hilbert space L_i^2, there exist a unique (up to unitary equivalence) decomposition $L_i^2 = \oplus_k L_i^2(\varphi_i^k)$, where φ_i^1 is maximal in L_i^2 relatively to T_i, and a decreasing set of multiplicity sets e_k^i, where e_1^i is the whole line, such that $\oplus_k L_i^2(\varphi_i^k)$ is equivalent with $\oplus_k L^2(e_k^i, \mu_i)$, where the measure of the ordered representation is defined as $\mu_i(\cdot) = (E^i(\cdot)\varphi_i^1, \varphi_i^1)_i$. A spectral representation of T_i in $\oplus_k L^2(e_k^i, \mu_i)$ is called the *ordered representation* and it is unique, up to a unitary equivalence. Two operators are called *equivalent*, if they create the same ordered representation of their spaces.

A well-known theorem [18, Ch. XIII, Section 5, Theorem 1]) represents the structural result for the ordered representation of the operator T_i in its abstract form. Since the generalized eigenfunctions $W_k(x, \lambda)$ from this theorem are only measurable with respect to the spectral parameter λ, the usual technique is to decompose them using an analytical basis of solutions of the equation $(\tau_i - \lambda)\sigma = 0$. At that, frequently we do not need all the basis functions and use only a part of them. The *Defining system* $\sigma_1, \ldots, \sigma_s$ is the subsystem of the solution basis such that all $W_k(\cdot, \lambda)$ belong to its linear capsule. This treatment leads to a very important conception of matrix Hilbert spaces.

2. The structure of the spectral representation for the vector-operator T

For $i \in \Omega$, we introduce a *sliced union* of sets M_i (see also preliminaries) as a set M, containing all M_i on different copies of $\cup_{i\in\Omega} M_i$. The sets M_i do not intersect in M, but they can *superpose*, i.e. two sets M_i and M_j superpose, if their projections in the set $\cup_{i\in\Omega} M_i$ intersect.

For $z_i \in L_i^2$, $i \in \Omega$, define $\widehat{\mathbf{z_i}} = \{0, ..., 0, z_i, 0, ..., 0\} \in \mathbf{L}^2$, where z_i is on the i-th place.

For each $i \in \Omega$, let $\epsilon(T_i)$ denote the *subspectrum* of the operator T_i, i.e. the set where the spectral measures of T_i are concentrated. Note that $\overline{\epsilon(T_i)} = \sigma(T_i)$. For instance, the subspectrum of an operator having the complete system of eigenfunctions with eigenvalues being the rational numbers of $[0, 1]$ equals to $\mathbb{Q} \cap [0, 1]$; the subspectrum of an operator having the continuous spectrum [0,1] is assumed to equal to (0,1) without loss of generality. For this see also [17, Chapter VII.2].

Consider a projecting mapping $P : M \to \cup_{i\in\Omega} M_i$ such that $P(\epsilon(T_i)) = \epsilon(T_i)$.

Let $\Omega = \cup_{k=1}^K A_k$, $A_k \cap A_s = \emptyset$ for $k \neq s$ and

$$A_k = \{s \in \Omega : \forall s, l \in A_k, s \neq l, P(\epsilon(T_s)) \cap P(\epsilon(T_l)) = B_{sl},$$
$$\text{where } \|E^t(B_{sl})\varphi_t\|_t^2 = 0 \text{ for any cyclic } \varphi_t \in L_t^2,\ t = s, l\}.$$

From all such divisions of Ω we choose and fix the one, which contains the minimal number of A_k. In the case when all the coordinate spectra $\sigma(T_i)$ are simple, we define the number $\Lambda = \min\{K\}$ as the *spectral index* of the vector-operator T.

In [12], the following two lemmas were proved:

Lemma 2.1. *The identity resolution $\{E_\lambda\}$ of the vector-operator T equals to the direct sum of the coordinate identity resolutions $\{E_\lambda^i\}$, that is $\{E_\lambda\} = \oplus_{i\in\Omega}\{E_\lambda^i\}$*

Lemma 2.2. *Let each T_i have a cyclic vector a_i in L^2_i. Then the vector-operator T has minimum Λ cyclic vectors $\{\mathbf{a}_k\}_{k=1}^{\Lambda}$, having the form $\mathbf{a}_k = \sum_{i\in A_k} \widehat{\mathbf{a}_\mathbf{i}}$.*

Below we present the three theorems (2.3, 2.4 and 2.5) without their complete proofs. Only the structural parts of the proofs essential for the current study are presented. The complete proof of Theorem 2.3 may be found in [14, 13] and the reader can refer to [15] for the proofs of Theorems 2.4 and 2.5.

Theorem 2.3. *If θ_i and $\{e^i_n\}_{n=1}^{m_i}$ are measures and multiplicity sets of ordered representations for coordinate operators T_i, $i \in \Omega$, then there exist processes Pr_1 and Pr_2, such that the measure*

$$\theta = Pr_1(\{\theta_i\}_{i\in\Omega})$$

is the measure of an ordered representation and the sets

$$s_n = Pr_2\left(\{e^i_k\}_{i\in\Omega;\, k=\overline{1,m_i}}\right)$$

are the canonical multiplicity sets of the ordered representation of the operator T. Thus, the unitary representation of the space $\mathbf{L}^2$ on the space $\oplus_n L^2(s_n, \theta)$ is the ordered representation and it is unique up to unitary equivalence.

Proof. We divide the proof into units for convenience. Parts **(A)** and **(B)** represent the process, which we call 'the process of division on subspectra'.

(A) Let a_i be maximal vectors relative to the operators T_i in L^2_i. We want to find a maximal vector relative to the vector-operator T. We know, that the vector $\oplus_{i\in\Omega} a_i$ does not give a single measure, if a set $P(\epsilon(T_i)) \cap P(\epsilon(T_j))$ has a non-zero spectral measure for $i \neq j$. Consider restrictions $T_i|_{L^2_i(a_i)} = T'_i$. Since all the operators T'_i have single cyclic vectors a_i, we can divide Ω into A_k, $k = \overline{1, \Lambda}$ and apply Lemma 2.2 for the operator $\oplus_{i\in\Omega} T'_i$. Then we derive Λ vectors $\mathbf{a}^k = \oplus_{j\in A_k} a_j$, which are maximal in the respective spaces $\mathbf{L}^2(\mathbf{a}^k) = \oplus_{j\in A_k} L^2_j(a_j)$.

(B) Now let $1 < \Lambda < \infty$. Define $T^k = \oplus_{j\in A_k} T'_j$. For any two operators T^k and T^s, $k \neq s$, let us introduce the sets $\epsilon_{k,s} = P(\epsilon(T^k)) \cap P(\epsilon(T^s))$ and $\epsilon_k = P(\epsilon(T^k)) \setminus \epsilon_{k,s}$. There exist unitary representations

$$U^k : \mathbf{L}^2(\mathbf{a}^k) \to L^2(\mathbb{R}, \mu_{\mathbf{a}^k}).$$

Consider measures μ_k and $\mu_{k,s}$, defined as

$$\mu_{k,s}(e) = \mu_{\mathbf{a}^k}(e \cap \epsilon_{k,s})$$

and $\mu_k(e) = \mu_{\mathbf{a}^k}(e \cap \epsilon_k)$, for any measurable set e. For any operator T^k (with respect to T^s), measures μ_k and $\mu_{k,s}$ are mutually singular and $\mu_k + \mu_{k,s} = \mu_{\mathbf{a}^k}$; therefore

$$L^2(\mathbb{R}, \mu_{\mathbf{a}^k}) = L^2(\mathbb{R}, \mu_k) \oplus L^2(\mathbb{R}, \mu_{k,s}).$$

This means that (according to our designations):

$$U^{k^{-1}} : L^2(\mathbb{R}, \mu_{\mathbf{a}^k}) \longrightarrow \mathbf{L}^2(\mathbf{a}^k_k) \oplus \mathbf{L}^2(\mathbf{a}^k_{k,s})$$

and

$$\mathbf{a}^k = \mathbf{a}^k_k \oplus \mathbf{a}^k_{k,s}, \tag{2}$$

where $\mathbf{a}^k_k$ and $\mathbf{a}^k_{k,s}$ form the measures μ_k and $\mu_{k,s}$ respectively. Define also as $\max\{w,\psi\}$ the vector, which is maximal of the two vectors in the brackets (Note that this designation is valid only for vectors, considered on the same set. In order not to complicate the investigation we assume here that any two vectors are comparable in this sense. In order to achieve this, it is enough to decompose each coordinate operator T_i into the direct sum $T_i^{pp} \oplus T_i^{cont}$, where the operators have respectively pure point and continuous spectra. Then after redesignation we obtain the equivalent vector-operator to the initial vector-operator $\oplus T_i$).

Consider first two operators T^1 and T^2. It is clear, that the vector

$$\mathbf{a}^{1\oplus 2} = \mathbf{a}^1_1 \oplus \mathbf{a}^2_2 \oplus \max\left\{\mathbf{a}^1_{1,2}, \mathbf{a}^2_{2,1}\right\}$$

is maximal in $\mathbf{L}^2(\mathbf{a}^1) \oplus \mathbf{L}^2(\mathbf{a}^2)$. Note that $\mathbf{a}^1_1$ and $\mathbf{a}^2_2$ and they both may equal zero. The maximal vector in $\mathbf{L}^2(\mathbf{a}^1) \oplus \mathbf{L}^2(\mathbf{a}^2) \oplus \mathbf{L}^2(\mathbf{a}^3)$ will have the form:

$$\mathbf{a}^{1\oplus 2\oplus 3} = \mathbf{a}^{1\oplus 2}_{1\oplus 2} \oplus \mathbf{a}^3_3 \oplus \max\left\{\mathbf{a}^{1\oplus 2}_{1\oplus 2,3}, \mathbf{a}^3_{3,1\oplus 2}\right\},$$

where $\mathbf{a}^{1\oplus 2}_{1\oplus 2}$ is the narrowed vector $\mathbf{a}^{1\oplus 2}$, corresponding to the set which is free from the superposition with $\epsilon(T_3)$, as shown in (2).

Continuing this process, we obtain a maximal vector in the main space $\mathbf{L}^2$:

$$\mathbf{a}^{1\oplus\cdots\oplus\Lambda} = \mathbf{a}^{1\oplus\cdots\oplus\Lambda-1}_{1\oplus\cdots\oplus\Lambda-1} \oplus \mathbf{a}^\Lambda_\Lambda \oplus \max\left\{\mathbf{a}^{1\oplus\cdots\oplus\Lambda-1}_{1\oplus\cdots\oplus\Lambda-1,\Lambda}, \mathbf{a}^\Lambda_{\Lambda,1\oplus\cdots\oplus\Lambda-1}\right\}. \tag{3}$$

Let $\Lambda = \infty$. We obtain $\mathbf{a}^{1\oplus\cdots\oplus\Lambda}$ as a vector which satisfies the following equality:

$$\left\|\left[\oplus_{i\in\Omega} E^i(\cdot)\right]\mathbf{a}^{1\oplus\cdots\oplus\Lambda}\right\|^2 = \lim_{L\to\infty}\left\|\left[\oplus_{j=1}^L E^j(\cdot)\right]\mathbf{a}^{1\oplus\cdots\oplus L}\right\|^2, \tag{4}$$

since the limit on the right side exists.

(C) The next step is to build the measure of the ordered representation for the vector-operator. From Lemma 2.1 and the reasonings above, it follows that such a measure will be

$$\theta(\cdot) = \left(\left[\oplus_{i\in\Omega} E^i(\cdot)\right]\mathbf{a}^{1\oplus\cdots\oplus\Lambda}, \mathbf{a}^{1\oplus\cdots\oplus\Lambda}\right).$$

(D) The canonical multiplicity sets s_n of the vector-operator have the form:

$$s_n = \left[\bigcup_i P(e^i_n)\right] \bigcup \left[\bigcup_{\sum m_i \geqslant n} \bigcap P\left(e^i_{m_i} \backslash e^i_{m_i+1}\right)\right]. \tag{5}$$

□

Let $I = \bigvee_{i\in\Omega} I_i$ denote the sliced union of intervals I_i. Similarly, $I^k = \bigvee_{j\in A_k} I_j$. If x_i are variables on I_i, then $\vee x_i$ will designate a variable either on I or I^k depending on the context. This notation shows, that a vector-function

$$z = \{z_1(x_1), \dots, z_n(x_n), \dots\}$$

on I or I^k may be written as $z(\vee x_i)$. In particular, we may also write $\mathbf{z}(\vee x_i)$ instead of $\mathbf{z} = \oplus_{i\in\Omega} z_i$.

Let us introduce the space $\oplus_{i\in\Omega}L^\infty(I_i^n)$. Here, $\mathbf{z}(\vee x_i)\in\oplus_{i\in\Omega}L^\infty(I_i^n)$ means that

$$\sup_{i\in\Omega}\left\{\operatorname{ess}\sup_{x_i\in I_i^n}|z_i(x_i)|\right\}<\infty,$$

where for each i, families $\{I_i^n\}_{n=1}^\infty$ represent compact subintervals of I_i, such that $\cup_{n=1}^\infty I_i^n=I_i$. In [4, Lemma 2.1], it was shown that $\oplus_{i\in\Omega}L^\infty(I_i^n)=(\oplus_{i\in\Omega}L^1(I_i^n))^*$, where the space of Lebesgue-integrable vector-functions $\oplus_{i\in\Omega}L^1(I_i^n)$ is defined analogously to $\mathbf{L}^2$.

We also need to introduce a symbolic integral $\int_{\bigvee J_i}f(\vee x_i)\,d(\vee x_i)$ defined by:

$$\int_{\bigvee J_i}f(\vee x_i)\,d(\vee x_i)=\oplus_i\int_{J_i}f_i(x_i)\,dx_i,$$

where $f(\vee x_i)$ is understood to be measurable relatively to $d(\vee x_i)$, if and only if $f_i(x_i)$ are measurable relative to Lebesgue measures dx_i. Then

$$\int_{\bigvee J_i}f(\vee x_i)\,d(\vee x_i)<\infty$$

if and only if $\sup_i\int_{J_i}f_i(x_i)\,dx_i<\infty$.

Theorem 2.4. *Let T be a self-adjoint vector-operator, generated by an EMZ system $\{I_i,\tau_i\}_{i\in\Omega}$. Let U be an ordered representation of the space $\mathbf{L}^2=\oplus_{i\in\Omega}L^2(I_i)$ relative to T with the measure θ and the multiplicity sets s_k, $k=\overline{1,m}$. Then there exist kernels $\Theta_k(\vee x_i,\lambda)$, measurable relative to $d(\vee x_i)\times\theta$, such that $\Theta_k(\vee x_i,\lambda)=0$ for $\lambda\in\mathbb{R}\setminus s_k$ and $(\oplus_{i\in\Omega}\tau_i-\lambda)\Theta_k(\vee x_i,\lambda)=0$ for each fixed λ. Moreover for any bounded Borel set Δ,*

(6)
$$\int_\Delta|\Theta_k(\vee x_i,\lambda)|^2\,d\theta(\lambda)\in\oplus_{i\in\Omega}L^\infty(I_i^n)\ \ \forall n\in\mathbb{N}.$$

(7)
$$(U\mathbf{w})^k(\lambda)=\lim_{n\to\infty}\int_{I^n}\mathbf{w}(\vee x_i)\,\overline{\Theta_k(\vee x_i,\lambda)}\,d(\vee x_i),\quad \mathbf{w}\in\mathbf{L}^2,$$

where the limit exists in $L^2(s_k,\theta)$. The kernels $\{\Theta_k(\vee x_i,\lambda)\}_{k=1}^n$, $n\leqslant m$, are linearly independent as vector-functions of the first variable almost everywhere relative to the measure θ on s_n.

Proof. Fix i. If θ_i and $\{e_p^i\}_{p=1}^{m_i}$ are respectively the measure and the multiplicity sets of an ordered representation for T_i, then there exists the decomposition $L_i^2=\oplus_{p=1}^{m_i}L^2(e_p^i,\theta_i)$, which implies $T_i=\oplus_{p=1}^{m_i}T_i^p$ and $L^2(e_p^i,\theta_i)$ are T_i^p-invariant. For vector-operator $(\oplus_{i\in\Omega}\oplus_{p=1}^{m_i}T_i^p)\to$ redesignate $\to\oplus_sT_s$, $s=\{i,p\}\in\Omega_1$, we may write $\Omega_1=\cup_{k=1}^\Lambda A_k$.

Let us separate the proof into units for convenience.

(A) For each T_j, $j\in A_k$ and $k=\overline{1,\Lambda}$, there exists a single cyclic vector $a_j\in L_j^2$ and [18, XII.3, Lemma 9 and XIII.5, Theorem 1(I)] a function $W_j(x_j,\lambda)$ defined on $I_j\times e_j$ (note, that for a fixed $i\in\Omega$, $I_j=I_i$ for all $p=\overline{1,m_i}$) and measurable relative to $dx_j\times\mu_{a_j}$, such that $W_j(x_j,\lambda)=0$, $\lambda\in\mathbb{R}\setminus e_j$ and for any bounded $\Delta\subset e_j$: $\int_\Delta|W_j(x_j,\lambda)|^2\,d\mu_{a_j}(\lambda)\in L^\infty(I_j^n)$, $n\in\mathbb{N}$. Also

(8)
$$\left(E^j(\Delta)F_j(T_j)a_j\right)(x_j)=\int_\Delta W_j(x_j,\lambda)F_j(\lambda)\,d\mu_{a_j}(\lambda),$$

for any $F_j \in L^2(e_j, \mu_{a_j})$. On $I^k = \bigvee_{j\in A_k} I_j$, we construct the vector-function

$$W^k(\vee x_j, \lambda) = \{W_1(x_1,\lambda), \dots, W_n(x_n,\lambda), \dots\},$$

which is obviously measurable relative to $d(\vee x_j) \times \sum \mu_{a_j}$. Separate arguments show that this vector-function is a correctly constructed generalized eigenfunction and thus satisfies the statement of the theorem within each A_k.

Note that since for all $p = \overline{1, m_i}$ there exists the equality $(\tau_i - \lambda)W_i^p = 0$ (see [18, XIII.5, Theorem 1]), it is obvious that $(\oplus_{j\in A_k}\tau_j - \lambda)W^k = 0$, where $\tau_j = \tau_i$ for a fixed i and all $p = \overline{1, m_i}$. If $P(\epsilon(T_i)) \cap P(\epsilon(T_j))$ has zero spectral measures for all $i, j \in \Omega$, then $A_k : \Omega_1 = \cup_{k=1}^{\Lambda_1} A_k$ may be constructed such that A_k contains of indices $\{i, k\}$, $i \in \Omega$, $k = \overline{1, \max_i\{m_i\}}$.

(B) Consider the set of indices $\Omega_2 = \{j \in \Omega_1 : j = \{i, 1\}, i \in \Omega\}$. Construct $A_k : \Omega_2 = \cup_{k=1}^{\Lambda_2} A_k$. Apply the reasonings used in **(A)**, considering everywhere Ω_2 instead of Ω_1. Hence, for each A_k and we find a vector-function $W_1^k(\vee x_j, \lambda)$ which is the solution of the equation $(\oplus_{j\in A_k}\tau_j - \lambda)\mathbf{y} = 0$. Consider W_1^k and W_1^s for $s \neq k$. For $\mathbf{a}^k$ there exists the decomposition $\mathbf{a}^k = \mathbf{a}_k^k \oplus \mathbf{a}_{k,s}^k$ (see the proof of Theorem 2.3). This fact induces the decomposition for W_1^k: $W_1^k = W_{1,k}^k \oplus W_{1,k,s}^k$. It is clear that being the restrictions of W_1^k, the vector-functions $W_{1,k}^k$ and $W_{1,k,s}^k$ are also the solutions of the equation $(\oplus_{j\in A_k}\tau_j - \lambda)\mathbf{y} = 0$. They, along with $\mathbf{a}_k^k$ and $\mathbf{a}_{k,s}^k$, define unitary transformations U_k^k and $U_{k,s}^k$, such that: $U_k^k : \mathbf{L}^2(\mathbf{a}_k^k) \to L^2(\mathbb{R}, \mu_k)$ and $U_{k,s}^k : \mathbf{L}^2(\mathbf{a}_{k,s}^k) \to L^2(\mathbb{R}, \mu_{k,s})$ (see the proof of Theorem 2.3). This implies, that the decomposition $W^k = W_{1,k}^k \oplus W_{1,k,s}^k$ is correct.

Define as $\max\{W_{1,k,s}^k, W_{1,s,k}^s\}$ the vector-function, which corresponds to the vector $\max\{\mathbf{a}_{k,s}^k, \mathbf{a}_{s,k}^s\}$, respectively $\min\{W_{1,k,s}^k, W_{1,s,k}^s\}$ as the vector-function which corresponds to that $\mathbf{a}_{k,s}^k$ or $\mathbf{a}_{s,k}^s$, which is not maximal of the two.

(C) Without loss of generality, suppose that $k = 1$ and $s = 2$. From the reasonings presented in Part **(A)** of this proof, it follows that

$$\Theta_1^{1\oplus 2} = W_{1,1}^1 \oplus W_{1,2}^2 \oplus \max\left\{W_{1,1,2}^1, W_{1,2,1}^2\right\}$$

is correctly constructed vector-function satisfying the statement of the theorem for the case $T = [\oplus_{j\in A_1} T_j] \oplus [\oplus_{q\in A_2} T_q]$. Apply the above described process to $\Theta_1^{1\oplus 2}$ and W_1^3 to obtain the correctly constructed vector-function:

$$\Theta_1^{1\oplus 2\oplus 3} = \Theta_{1,1\oplus 2}^{1\oplus 2} \oplus W_{1,3}^3 \oplus \max\left\{\Theta_{1,1\oplus 2,3}^{1\oplus 2}, W_{1,3,1\oplus 2}^3\right\}.$$

Continuing this process, we finally obtain:

$$\Theta_1(\vee x_i, \lambda) = \Theta_1^{1\oplus\cdots\oplus\Lambda_2} = \Theta_{1,1\oplus\cdots\oplus\Lambda_2-1}^{1\oplus\cdots\oplus\Lambda_2-1} \oplus W_{1,\Lambda_2}^{\Lambda_2} \oplus \max\left\{\Theta_{1,1\oplus\cdots\oplus\Lambda_2-1,\Lambda_2}^{1\oplus\cdots\oplus\Lambda_2-1}, W_{1,\Lambda_2,1\oplus\cdots\oplus\Lambda_2-1}^{\Lambda_2}\right\},$$

where in the case of $\Lambda_2 = \infty$, $\Theta_1^{1\oplus\cdots\oplus\Lambda_2}$ is the function which satisfies (analogously to (4)):

$$\left\|[\oplus_{i\in\Omega}E^i(\Delta)]\int_\Delta \Theta_1^{1\oplus\cdots\oplus\Lambda_2} d\theta(\lambda)\right\|^2 = \lim_{L\to\infty}\left\|[\oplus_{j=1}^L E^j(\Delta)]\int_\Delta \Theta_1^{1\oplus\cdots\oplus L}\, d\theta_L(\lambda)\right\|^2, \tag{9}$$

for any bounded Borel set Δ, where

$$\theta_L(\cdot) = \left([\oplus_{j=1}^L E^j(\cdot)]\,\mathbf{a}^{1\oplus\cdots\oplus L}, \mathbf{a}^{1\oplus\cdots\oplus L}\right)$$

is the measure of the ordered representation of the space $\oplus_{j=1}^{L} L_j^2$. The limit on the right side exists and in fact it appears that

$$\int_\Delta \Theta_1^{1\oplus\cdots\oplus L}\, d\theta_L(\lambda) \to \int_\Delta \Theta_1^{1\oplus\cdots\oplus\Lambda_2} d\theta(\lambda),$$

as $L \to \infty$.

(D) Define $\Omega_3 = \{j \in \Omega_1 : j = \{i, 2\}, i \in \Omega\}$. Construct $A_k : \Omega_3 = \cup_{k=1}^{\Lambda_3} A_k$. Apply processes **(B)** and **(C)** of this proof, substituting everywhere Ω_3 instead of Ω_2. We obtain a vector-function $\Theta_2^{1\oplus\cdots\oplus\Lambda_3}$, which is defined on the set $\cup_i P(e_2^i)$. But, as we know (see (5)), the set s_2 also includes the sets where there are non-empty superpositions of $\epsilon(T_i)$. Therefore, designating

$$\begin{aligned} \Theta_2^1 &= \Theta_2^{1\oplus\cdots\oplus\Lambda_3},\ \Theta_2^2 = \min\{W_{1,1,2}^1, W_{1,2,1}^2\}, \ldots, \\ \Theta_2^{\Lambda_2+1} &= \min\left\{\Theta_{1,1\oplus\cdots\oplus\Lambda_2-1,\Lambda_2}^{1\oplus\cdots\oplus\Lambda_2-1}, W_{1,\Lambda_2,1\oplus\cdots\oplus\Lambda_2-1}^{\Lambda_2}\right\}, \end{aligned}$$

we may again use the process **(C)** to build the vector-function $\Theta_2(\vee x_i, \lambda)$ defined on s_2 and $\Theta_2(\vee x_i, \lambda) = 0$ for $\lambda \in \mathbb{R} \setminus s_2$. Using processes **(B)**, **(C)**, **(D)** and formula (5), we finally obtain $\Theta_m(\vee x_i, \lambda)$.

(E) The linear independence is proved by separate arguments. □

Theorem 2.5 (Eigenfunction expansions)**.** *For any* $\mathbf{w} \in \mathbf{L}^2$*, there exists a decomposition*

$$\mathbf{w} = \sum_{k=1}^{m} \lim_{n\to\infty} \int_{-n}^{+n} (U\mathbf{w})^k(\lambda)\Theta_k(\vee x_i, \lambda)\, d\theta(\lambda),$$

Theorem 2.6. *Let T be a self-adjoint vector-operator, generated by an EMZ system $\{I_i, \tau_i\}_{i\in\Omega}$. Let the measure θ and the sets $\{s_k\}_{k=1}^m$ be respectively a measure and multiplicity sets of an ordered representation of the space $\mathbf{L}^2 = \oplus_{i\in\Omega} L^2(I_i)$, relative to the operator T. The kernels $\{\Theta_k\}_{k=1}^m$ are the generalized vector-operator eigenfunctions, corresponding to the multiplicity sets (as defined in Theorem 2.4). Given a bounded Borel function F, which equals zero beyond a compact Borel set Δ, the bounded vector-operator $F(T)$ may be represented as an integral operator:*

$$[F(T)\mathbf{f}](\vee s_i) = \int_I \mathbf{f}(\vee x_i) K(F; \vee s_i, \vee x_i)\, d(\vee x_i), \tag{10}$$

where $\mathbf{f} \in \mathbf{L}^2$ *and*

$$K(F; \vee x_i, \vee s_i) = \sum_{k=1}^{m} \int_\Delta F(\lambda)\Theta_k(\vee x_i, \lambda)\overline{\Theta_k(\vee s_i, \lambda)}\, d\theta(\lambda). \tag{11}$$

Proof. From Theorem 2.5 it follows that

$$[F(T)\mathbf{f}](\vee x_i) = \sum_{k=1}^{m} \int_\Delta (UF(T)\mathbf{f})^k(\lambda)\Theta_k(\vee x_i, \lambda)\, d\theta(\lambda),$$

and since for any spectral representation

$$(UF(T)\mathbf{f})^k(\lambda) = F(\lambda)(U\mathbf{f})^k(\lambda),$$

from Theorem 2.4 we obtain:

$$\begin{aligned}(12)\qquad [F(T)\mathbf{f}](\vee s_i) &= \sum_{k=1}^{m}\int_\Delta F(\lambda)(U\mathbf{f})^k(\lambda)\Theta_k(\vee x_i,\lambda)\,d\theta(\lambda) = \\ &= \sum_{k=1}^{m}\int_\Delta F(\lambda)\Theta_k(\vee s_i,\lambda)\int_I \mathbf{f}(\vee x_i)\,\overline{\Theta_k(\vee x_i,\lambda)}\,d(\vee x_i)\,d\theta(\lambda),\end{aligned}$$

where

$$\int_I \mathbf{f}(\vee x_i)\,\overline{\Theta_k(\vee x_i,\lambda)}\,d(\vee x_i) = \lim_{n\to\infty}\int_{I^n}\mathbf{f}(\vee x_i)\,\overline{\Theta_k(\vee x_i,\lambda)}\,d(\vee x_i),$$

for which see formula (7).

Note that (see [18, XII.3.8]) $F(T)\mathbf{f}\in\oplus_{i\in\Omega}\left(\cap_{n=1}^\infty D(T_i^n)\right)$. For any system $\{J_i\}_{i\in\Omega}$ of compact subintervals of the respective intervals from $\{I_i\}_{i\in\Omega}$, define the space $\oplus_{i\in\Omega}C(J_i)=C(J)$, $J=\bigvee_i J_i$, as the space of continuous vector-functions with the norm

$$\|\mathbf{f}\|_{C(J)}=\sup_i\sup_{s_i\in J_i}|f_i(s_i)|.$$

Hence, the mapping $\mathbf{f}\to F(T)\mathbf{f}$ is continuous as the operator from $\mathbf{L}^2$ to $C(J)$. This means that there exists a constant $M(J)$, such that

$$\|F(T)\mathbf{f}\|_{C(J)}\leqslant M(J)\|\mathbf{f}\|_{\mathbf{L}^2},$$

or

$$(13)\qquad \sup_i\sup_{s_i\in J_i}|(F(T_i)f_i)(s_i)|\leqslant M(J)\|\mathbf{f}\|_{\mathbf{L}^2}.$$

Let $m<\infty$. For each $i\in\Omega$, define $\mathcal{H}_i$ as a dense set in L_i^2 consisting of functions equalling zero beyond a compact subset of I_i. We can interchange the integrals in (12):

$$(14)\qquad [F(T)\mathbf{f}](\vee s_i)=\int_I \mathbf{f}(\vee x_i)K(F,\vee x_i,\vee s_i)\,d(\vee x_i),$$

for $\mathbf{f}\in\oplus_{i\in\Omega}\mathcal{H}_i$ and

$$K(F;\vee x_i,\vee s_i)=\sum_{k=1}^m\int_\Delta F(\lambda)\Theta_k(\vee x_i,\lambda)\overline{\Theta_k(\vee s_i,\lambda)}\,d\theta(\lambda).$$

From (13) we obtain:

$$\sup_i\sup_{s_i\in J_i}\left|\int_I\mathbf{f}(\vee x_i)K(F,\vee x_i,\vee s_i)\,d(\vee x_i)\right|\leqslant M(J)\|\mathbf{f}\|_{\mathbf{L}^2}.$$

It is now clear, that the formula (14) holds for any $\mathbf{f}\in\mathbf{L}^2$.

Pass now to the case $m=\infty$. Recall that

$$U:\mathbf{L}^2\to\oplus_{k=1}^\infty L^2(s_k,\theta).$$

For each $n<\infty$ define an orthogonal projector $P_n:\mathbf{L}^2\to\mathbf{L}^2$, such that

$$P_n\oplus_{i\in\Omega}f_i=\{f_1,\dots,f_n,0,0,\dots\}.$$

Define continuous linear functionals $\phi_{s_i}(f_i)=(F(T_i)f_i)(s_i)$, for which there exist $g_{s_i}\in L_i^2$ such that $(F(T_i)f_i)(s_i)=(f_i,g_{s_i})_i$. From the reasonings presented in the beginning of this proof for the case of the finite multiplicity, we obtain:

$$\begin{aligned}(F(T)U^{-1}P_nU\mathbf{f})(\vee s_i) &= (U^{-1}P_nUF(T)\mathbf{f})(\vee s_i) = \\ &= \sum_{k=1}^{n} \int_\Delta F(\lambda)\Theta_k(\vee s_i,\lambda) \int_I \mathbf{f}(\vee x_i)\, \overline{\Theta_k(\vee x_i,\lambda)}\, d(\vee x_i)\, d\theta(\lambda).\end{aligned}$$

That is

$$(F(T)U^{-1}P_nU\mathbf{f})(\vee s_i) = \int_I \mathbf{f}(\vee x_i) K_n(F; \vee x_i, \vee s_i)\, d(\vee x_i), \tag{15}$$

where

$$K_n(F; \vee x_i, \vee s_i) = \sum_{k=1}^{n} \int_\Delta F(\lambda)\Theta_k(\vee x_i,\lambda)\overline{\Theta_k(\vee s_i,\lambda)}\, d\theta(\lambda).$$

Since

$$\begin{aligned}&(F(T)U^{-1}P_nU\mathbf{f})(\vee s_i) = \oplus_{i\in\Omega}(F(T_i)[U^{-1}P_nU]_i f_i)(s_i) = \\ &\quad = \left\{([U^{-1}P_nU]_1 f_1, g_{s_1})_1, ([U^{-1}P_nU]_2 f_2, g_{s_2})_2, \ldots, ([U^{-1}P_nU]_j f_j, g_{s_j})_j, \ldots\right\} = \\ &\qquad = \left\{(f_1, [U^{-1}P_nU]_1 g_{s_1})_1, (f_2, [U^{-1}P_nU]_2 g_{s_2})_2, \ldots, (f_j, [U^{-1}P_nU]_j g_{s_j})_j, \ldots\right\},\end{aligned}$$

since the coordinate of a unitary vector-operator is unitary in the coordinate space. From this formula and (15) we obtain that the coordinate $K_n^i(F; x_i, \cdot)$ of $K_n(F; \vee x_i, \cdot)$ satisfies the equation

$$\overline{K_n^i(F; x_i, s_i)} = [U^{-1}P_nU]_i g_{s_i}.$$

From the last equality it follows that

$$\lim_{n\to\infty} \overline{K_n^i(F; x_i, s_i)} = \lim_{n\to\infty} [U^{-1}P_nU]_i g_{s_i},$$

and thus $\overline{K^i(F; x_i, s_i)} = g_{s_i}$ and $\overline{K(F; \vee x_i, \vee s_i)} = \oplus_{i\in\Omega} g_{s_i}$ which means that the series defining $K(F; x_i, s_i)$ converges in $\mathbf{L^2}$ for each fixed $\vee s_i$. Moreover,

$$\begin{aligned}\int_I \mathbf{f}(\vee x_i) K(F; \vee x_i, \vee s_i)\, d(\vee x_i) &= \oplus_{i\in\Omega} \int_I f_i(x_i) K^i(F; x_i, s_i)\, dx_i = \\ &= \left\{(f_1, g_{s_1})_1, (f_2, g_{s_2})_2, \ldots, (f_j, g_{s_j})_j, \ldots\right\} = \\ &= \oplus_{i\in\Omega}(F(T_i)f_i)(s_i) = (F(T)\mathbf{f})(\vee s_i).\end{aligned}$$

The theorem is proved. □

Since the kernels from Theorem 2.4 are only measurable relative to λ, the following theorem is important to strengthen the practical value of Theorems 2.4, 2.5 and 2.6:

Theorem 2.7. *Each kernel $\Theta_k(\vee x_i, \lambda)$, $k = \overline{1, m}$, may be decomposed as*

$$\Theta_k(\vee x_i, \lambda) = \sum_{s=1}^{M_k} \gamma_{sk}(\lambda)\sigma_{sk}(\vee x_i, \lambda), \tag{16}$$

where the M_k are finite for each k and $\sigma_{sk}(\vee x_i, \lambda)$ depend analytically on λ.

Proof. We separate the proof in parts which will correspond to the analogous parts of the proof of theorem 2.4.

(A') Each kernel $W_j(x_j,\lambda)$ from the part **(A)** of the proof of Theorem 2.4 may be decomposed:

$$W_j(\cdot,\lambda)=\sum_{s=1}^{n_j}\alpha_{js}(\lambda)\sigma_{js}(\cdot,\lambda),$$

where α_{js} are supposed to equal zero on $\mathbb{R}\setminus e_j$, see [18, p. 1351]. Supplementing the defining systems with zeros where necessary, we obtain:

$$\begin{aligned}W^k(\vee x_j,\lambda)=\oplus_{j\in A_k}W_j(x_j,\lambda) &= \oplus_{j\in A_k}\sum_{s=1}^{n_j}\alpha_{js}(\lambda)\sigma_{js}(x_j,\lambda)=\\ &= \sum_{q=1}^{N_k}\oplus_{j\in A_k}\alpha_{jq}(\lambda)\sigma_{jq}(x_j,\lambda)=\sum_{q=1}^{N_k}\alpha_q^k(\lambda)\sigma_q^k(\vee x_j,\lambda),\end{aligned}$$

where

$$N_k=\max_{j\in A_k}n_j,\ \alpha_q^k(\lambda)=\sum_{j\in A_k}\alpha_{jq}(\lambda)\text{ and }\sigma_q^k(\vee x_j,\lambda)=\oplus_{j\in A_k}\sigma_{jq}(x_j,\lambda).$$

Since e_j and e_k do not intersect almost everywhere for $j,k\in\Omega_2$, $j\neq k$, the series $\sum_{j\in A_k}\alpha_{jq}(\lambda)$ converges almost everywhere on $\cup_{j\in\Omega_2}P(e_j)$.

(C') Now pass to part **(B)**. There we obtained the decompositions $W_1^k=W_{1,k}^k\oplus W_{1,k,s}^k$ and $W_1^s=W_{1,s}^s\oplus W_{1,s,k}^s$. Let us totally order the set $\{T^j\}_{j=1}^{\Lambda_2}$ saying that $T^k\preceq T^s$ if $\max\{W_{1,k,s}^k,W_{1,s,k}^s\}=W_{1,k,s}^k$. At that, $T^k\simeq T^s$ if and only if $T^k\preceq T^s$ and $T^s\preceq T^k$. According to this, we build $\oplus_{j=1}^{\Lambda_2}T^j$, where $T^j\preceq T^{j+1}$, $j=\overline{1,\Lambda_2-1}$ if $\Lambda_2\geqslant 2$. The obtained vector-operator is obviously equivalent to the initial vector-operator (comprising unordered operators). Note that

$$W_1^k(\vee x_i,\lambda)=\sum_{q=1}^{N_k}\alpha_{1q}^k(\lambda)\sigma_{1q}^k(\vee x_j,\lambda)$$

and analogously

$$W_1^s(\vee x_i,\lambda)=\sum_{p=1}^{N_s}\alpha_{1p}^s(\lambda)\sigma_{1p}^s(\vee x_j,\lambda).$$

All the above leads to the following:

$$\begin{aligned}\Theta_1^{1\oplus 2} &= W_{1,1}^1\oplus W_{1,2}^2\oplus\max\left\{W_{1,1,2}^1,W_{1,2,1}^2\right\}=W_1^1\oplus W_{1,2}^2=\\ &= \left(\sum_{q=1}^{N_1}\alpha_{1q}^1(\lambda)\sigma_{1q}^1(\vee x_j,\lambda)\right)\oplus\left(\sum_{p=1}^{N_2}\alpha_{1p}^2(\lambda)\chi_{\epsilon_2}(\lambda)\sigma_{1p}^2(\vee x_j,\lambda)\right)=\\ &= \sum_{s=1}^{N^{1\oplus 2}}\alpha_{1s}^{1\oplus 2}(\lambda)\sigma_{1s}^{1\oplus 2}(\vee x_j,\lambda)\end{aligned}$$

where

$$N^{1\oplus 2}=\max\{N_1,N_2\};\ \alpha_{1s}^{1\oplus 2}(\lambda)=\alpha_{1s}^1(\lambda)+\alpha_{1s}^2(\lambda)\chi_{\epsilon_2}(\lambda),$$
$$\sigma_{1s}^{1\oplus 2}(\vee x_j,\lambda)=\sigma_{1s}^1(\vee x_j,\lambda)\oplus\left(\sigma_{1s}^2(\vee x_j,\lambda)\chi_{\epsilon_2}(\lambda)\right),\ s=\overline{1,N^{1\oplus 2}}.$$

Continuing this process till the finite Λ_2, we obtain:

$$\Theta_1(\vee x_i,\lambda)=\Theta_1^{1\oplus\cdots\oplus\Lambda_2}=\sum_{s=1}^{N^{1\oplus\cdots\oplus\Lambda_2}}\alpha_{1s}^{1\oplus\cdots\oplus\Lambda_2}(\lambda)\sigma_{1s}^{1\oplus\cdots\oplus\Lambda_2}(\vee x_j,\lambda), \tag{17}$$

where $N^{1\oplus\cdots\oplus\Lambda_2} = \max\{N_1, N_2, \dots N_{\Lambda_2}\}$ and for $s = \overline{1, N^{1\oplus\cdots\oplus\Lambda_2}}$:

$$\alpha_{1s}^{1\oplus\cdots\oplus\Lambda_2}(\lambda) = \alpha_{1s}^{1}(\lambda) + \sum_{i=2}^{\Lambda_2} \alpha_{1s}^{i}(\lambda)\chi_{\epsilon_i}(\lambda); \tag{18}$$

$$\sigma_{1s}^{1\oplus\cdots\oplus\Lambda_2}(\vee x_j, \lambda) = \sigma_{1s}^{1}(\vee x_j, \lambda) \oplus \left(\oplus_{i=2}^{\Lambda_2}\sigma_{1s}^{i}(\vee x_j, \lambda)\chi_{\epsilon_i}(\lambda)\right).$$

In the case of infinite Λ_2, $N^{1\oplus\cdots\oplus\Lambda_2}$ is clearly finite. The series in the right side of (18) pointwise converges, since it consists of items defined on non-intersecting sets. $\sigma_{1s}^{1\oplus\cdots\oplus\Lambda_2}(\vee x_j, \lambda)$ is defined by induction as the element which satisfies (see 9)

$$\lim_{L\to\infty}\left\|\int_\Delta \sigma_{1s}^{1\oplus\cdots\oplus\Lambda_2}(\vee x_j, \lambda)\, d\theta(\lambda) - \int_\Delta \sigma_{1s}^{1\oplus\cdots\oplus L}(\vee x_j, \lambda)\, d\theta_L(\lambda)\right\| = 0.$$

Since for each finite iteration the equality (17) is fulfilled, it is clear that for an infinite Λ_2 it will be fulfilled too.

(D') Borrowing the designations from **(D)** and using processes described in **(A')** and **(C')**, we shall come to the decomposition of $\Theta_2^{1\oplus\cdots\oplus\Lambda_3}$:

$$\Theta_2^{1\oplus\cdots\oplus\Lambda_3} = \sum_{s=1}^{N^{1\oplus\cdots\oplus\Lambda_3}} \alpha_{2s}^{1\oplus\cdots\oplus\Lambda_3}(\lambda)\sigma_{2s}^{1\oplus\cdots\oplus\Lambda_3}(\vee x_j, \lambda).$$

To obtain $\Theta_2(\vee x_i, \lambda)$, as in **(D)**, we repeat part **(C')** for

$$\Theta_2^1 = \Theta_2^{1\oplus\cdots\oplus\Lambda_3},\ \Theta_2^2 = W_{1,2,1}^2, \dots, \Theta_2^{\Lambda_2+1} = W_{1,\Lambda_2,1\oplus\cdots\oplus\Lambda_2-1}^{\Lambda_2}.$$

Finally, in the same way we obtain decompositions for all $\Theta_k(\vee x_i, \lambda)$, $k = \overline{1, m}$, which will have the form (16). □

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